Quantum Distributed Algorithms for Approximate Steiner Trees and Directed Minimum Spanning Trees

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NASA RIACS USRA Universities Space Research Association INTELLIGENT SYSTEMS DIVISION

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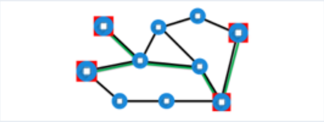
FEYNMAN QUANTUM ACADEMY - INTERNSHIP PROGRAM



Quantum Speedups for Distributed Graph Algorithms in the CONGEST-CLIQUE Model

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Years participated: 2022

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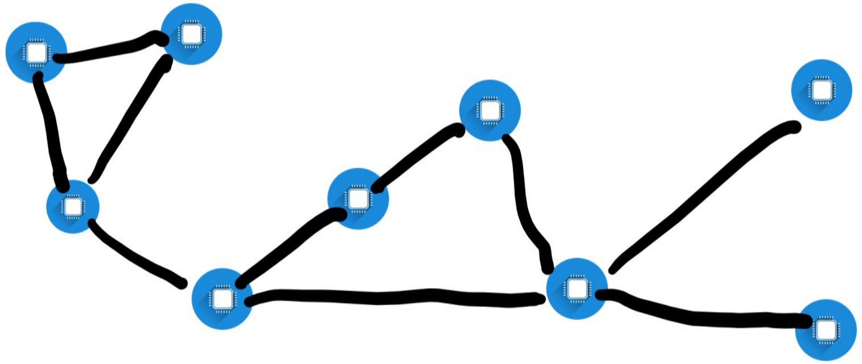
Distributed Computation: Network of Multiple Processors that communicate

Model as a graph where the processors are the nodes



- Example: Network of spacecraft, satellites, or control stations that each have computing power and can communicate
- **Distributed graph algorithms**: Answer some question about that graph, by computing across the processors in a distributed fashion

CONGEST Model:

1. Computation happens in *rounds* (compute, communicate, compute, communicate, …)
2. **Congested:** Communication limited by message size $\mathcal{O}(\log(n))$ **bits,** n being the number of nodes in the graph
3. Unlimited local computation at each node
4. Nodes can communicate only with their neighbors

CONGEST-CLIQUE Model:

1., 2., 3. above are the same

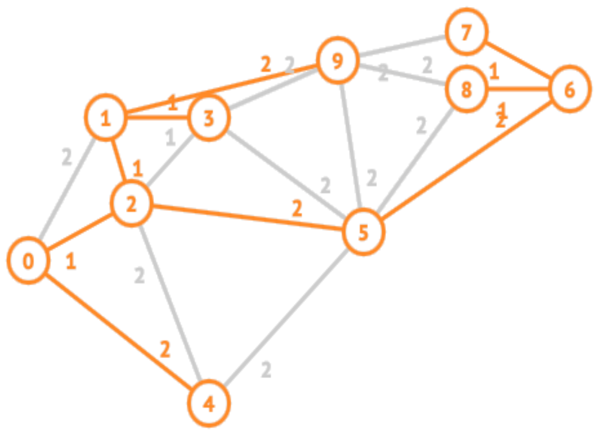
4. All nodes can communicate with one another

Introduction: Models

Initially: Each node knows its own ID and the ID's of its neighbors. Assume ID's are 1 to n → $\log(n)$ bits to encode

Answer a question about that graph in as few rounds as possible

- Ex: Spanning trees, subgraph detection, shortest paths...



A minimum spanning tree (orange) for the given graph (grey)

Quantum versions: Take CONGEST or CONGEST-CLIQUE, but allow messages to consist of $\mathcal{O}(\log(n))$ **qubits**

What problems can benefit from a quantum distributed method?

CONGEST: Negative Results

Main reference: Elkin et al 2012,

"Can Quantum Communication Speed Up Distributed Computation?"

- Proved limitations for quantum CONGEST model
- No speedups possible for many fundamental problems:
  Shortest paths, Minimum Spanning Tree, **Steiner Tree**, Min Cut, Hamiltonian Cycle…

- **Intuition:** In CONGEST, a significant bottleneck can be communicating between "distant" parts of the network – qubits don't help with that!

- Elkin, M., Klauck, H., Nanongkai, D., & Pandurangan, G. (2014, July). Can quantum communication speed up distributed computation?. In *Proceedings of the 2014 ACM symposium on Principles of distributed computing* (pp. 166-175).

CONGEST CLIQUE

- Elkin et al.'s results and analysis **do not** carry over to the CONGEST CLIQUE
- So, can quantum communication help in this model?

Surprising **positive results** in in Quantum CONGEST CLIQUE (QCC):

- Faster Triangle Detection, Izumi & Le Gall 2019
- Faster All-Pairs **Shortest-Paths** (APSP), Izumi & Le Gall 2020
  - $\tilde{O}(n^{1/4})$ in quantum versus $\tilde{O}(n^{1/3})$ in classical
  - This was in Elkin's list of problems not admitting speedups!

$$f(n) \in \tilde{O}(g(n))$$
if
$$\exists k: f(n) \in O(g(n) \log^k n)$$

Which other problems can exhibit speedups using the Quantum CONGEST CLIQUE model?

- Elkin, M., Klauck, H., Nanongkai, D., & Pandurangan, G. (2014, July). Can quantum communication speed up distributed computation?. In *Proceedings of the 2014 ACM symposium on Principles of distributed computing* (pp. 166-175).
- Izumi, T., & Le Gall, F. (2017, July). Triangle finding and listing in CONGEST networks. In Proceedings of the ACM Symposium on Principles of Distributed Computing (pp. 381-389).
- Izumi, T., & Le Gall, F. (2019, July). Quantum distributed algorithm for the All-Pairs Shortest Path problem in the CONGEST-CLIQUE model. In Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing (pp. 84-93).

Contributions

Algorithms in Quantum CONGEST-CLIQUE for **(approximately optimal) Steiner Trees** and **Directed Minimum Spanning Trees** that succeed with high probability

Computation using asymptotically fewer rounds than any known classical algorithm

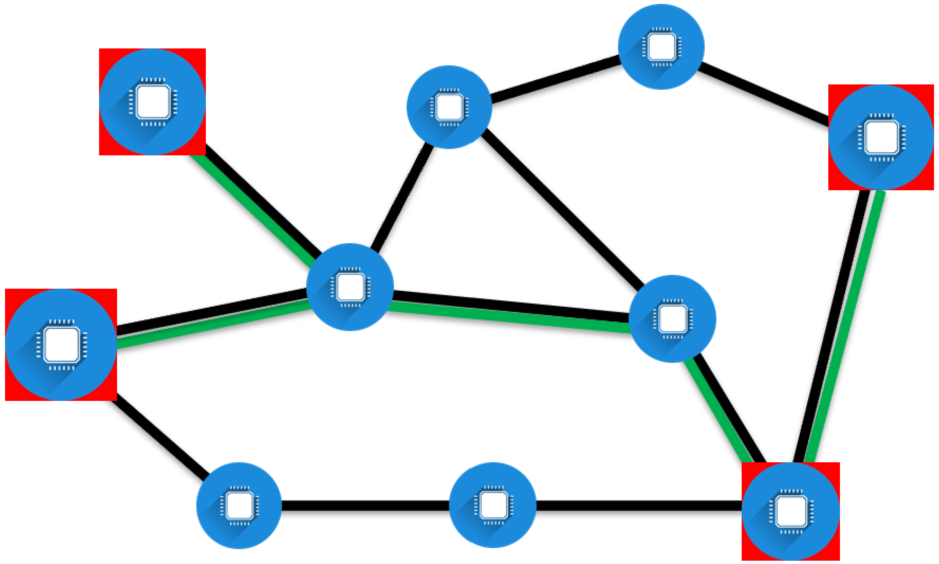
$\rightarrow \tilde{O}(n^{1/4})$ versus $\tilde{O}(n^{1/3})$

Exact complexity analysis of quantum and classical algorithms reveals

Improvements needed for both!

Steiner Tree Problem (informal):

Given a set of *terminal nodes*, find a minimum weight tree in the graph that contains them

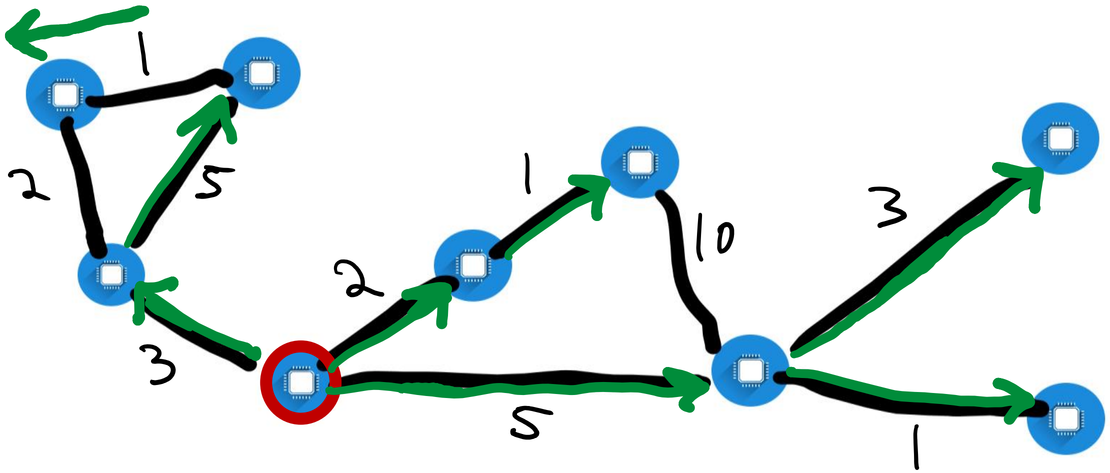


- Saikia, P., & Karmakar, S. (2020). Distributed approximation algorithms for steiner tree in the congested clique. *International Journal of Foundations of Computer Science*, *31*(07), 941-968.

Contributions

Directed Minimum Spanning Tree:

Given a root node **r** in a directed, weighted graph G, find a directed spanning tree for G rooted at r of minimum weight.



A Directed Minimum Spanning Tree (green) rooted at the red node

Techniques: Make use of fast all-pairs shortest-path with routing tables via triangle finding and distributed Grover Search to obtain the faster algorithms

Full Steiner Tree Approximation Algorithm

Step 1 - APSP and Routing Tables: Solve the APSP problem and add an efficient routing table scheme via triangle finding in $\tilde{O}(n^{1/4})$ rounds

Step 2 - Shortest-path Forest: Construct a shortest-path forest (SPF), where each tree consists of exactly one source terminal and the shortest paths to the vertices whose closest terminal is that source terminal. This step can be completed in one round and n messages

Step 3 - Weight Modifications: Modify the edge weights depending on whether they belong to a tree (set to 0), connect nodes in the same tree (set to ∞), or connect nodes from different trees (set to distance of shortest path between root terminals of the trees that use the edge).

Step 4 - Minimum Spanning Tree: Construct a minimum spanning tree (MST) on the modified graph in $O(1)$ rounds

Step 5 - Pruning: Prune leaves of the MST that are non-terminal nodes, since these are not needed for the Steiner Tree

Algorithmic Recipe & Complexity

1. Distributed Grover Search
   helps with…
2. Triangle Finding
   helps with…
3. Distance Products
   helps with…
4. Shortest Paths and Routing Tables
   helps with…
5. Steiner and Directed Minimum
   Spanning Trees!

$$\left.\begin{array}{l} \uparrow\ \boldsymbol{n^{\frac{1}{4}}\log(n)^3} \\ \uparrow\ \boldsymbol{\log(n)^3} \\ \uparrow\ \boldsymbol{\log(n)} \end{array}\right\} \boldsymbol{c\cdot n^{\frac{1}{4}}\log(n)^7} \text{ rounds}$$

In CONGEST CLIQUE, can solve anything in n rounds:
To be practical, need

$$\boldsymbol{3200\cdot n^{\frac{1}{4}}\log(n)^7 < n}$$

for which

$$\boldsymbol{n > 10^{20}}$$

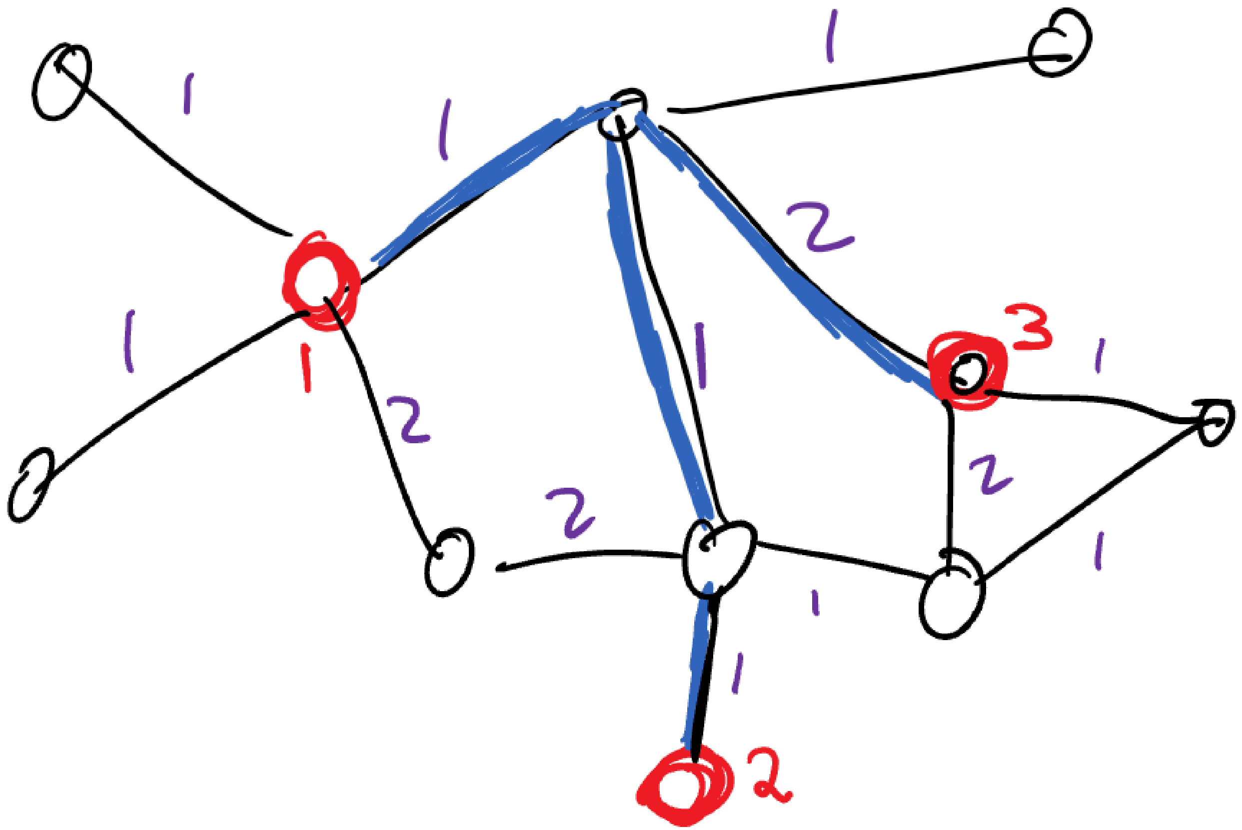
is required!! $\mathbf{10^{11}}$ for classical $\tilde{O}\left(n^{\frac{1}{3}}\right)$ counterpart.

Quantum Distributed Algorithms for Approximate Steiner Trees and Directed Minimum Spanning Trees

- We provide quantum distributed algorithms to tackle challenging graph problems
  - Approximate Stein Tree Problem
  - Directed Minimum Spanning Tree
- These quantum algorithms provide an asymptotic speedup with respect to the best known classical counterpart in terms of computational rounds in the CONGEST CLIQUE model
- We provided detailed analysis for the main algorithmic step: finding the all-pairs shortest paths
- We obtained complexity results realizing impractical scales where quantum counterparts become better than classical

Outline

Prune:



Future Work

Can improvements be made to have the asymptotic speedup be practical?

Search for more problems that can be sped up with similar techniques

- Make further use of all-pairs shortest paths or fast clique-finding? Speedups through distributed Grover search? Bounded-degree minimum spanning trees?

Generalize the Steiner Tree results?

Some existing algorithms with best $\tilde{O}$ complexity shown impractical -- others?

Distributed Implementation of Grover's Algorithm:

Suppose a set X of inputs, and a node u can evaluate a function $g: X \to \{0,1\}$ in r rounds. Then there exists a quantum distributed algorithm such that u outputs an element $x \in X$ with $g(x) = 1$ using $\mathcal{O}\left(r\sqrt{|X|}\right)$ rounds.

Uses the same ideas as centralized Grover search

Classical algorithm would use $\mathcal{O}(r|X|)$ rounds in absence of additional structure

Intuition: A node wants to inquire something related to some subset of other nodes – instead of inquiring about one node at a time, use superposition

Triangle Finding

Make use of Distributed Grover Search for finding Triangles!

Idea:

- Triangles live in $V \times V \times V$, so partition that space
- Have each node search an assigned partition for triangles
- Do this via sets $V_1 \times V_2 \times V_3$, where $|V_1| = |V_2| = n^{1/4}, |V_3| = n^{1/2}$, and they each $w \in V_1$ or V_2 contains $n^{3/4}$ nodes, and each $w \in V_3$ contains $n^{1/2}$ nodes.

- So $V_1 \times V_2 \times V_3$ has n elements that are each sets of nodes
  - each node gets one element
- Making use of fast routing, the bottleneck becomes searching through the elements of V_3
  - Grover can do it in $n^{1/4}$ instead of $n^{1/2}$

- Izumi, T., & Le Gall, F. (2017, July). Triangle finding and listing in CONGEST networks. In Proceedings of the ACM Symposium on Principles of Distributed Computing (pp. 381-389).

Distance Products

Definition 3.2. The *distance product* between two $n \times n$ matrices A and B, also sometimes referred to as the min-plus or tropical product, is defined as:

$$(A \star B)_{ij} = \min_k \{A_{ik} + B_{kj}\}. \quad (3.1)$$

- If W is the adjacency matrix of a graph, then the ij entry of $W \star W$ gives the shortest 2-hop path from node i to node j
- W^n contains shortest n-hop distances , so contains all shortest path distances (exponent with respect to $\star$) → take $\log(n)$ squares:

$$W^{2,\star} = W \star W, \;\; W^{4,\star} = \left(W^{2,\star}\right)^{2,\star}, \ldots, \;\; W^{2^{\lceil \log n \rceil},\star} = \left(W^{2^{\lceil \log n \rceil - 1},\star}\right)^{2,\star}$$

Distance Products via Triangles

Proposition 3.3. If $FindTriangleEdges^{-}$ on an n-node integer-weighted graph $G = (V, E, W)$ can be solved in $T(n)$ rounds, then the distance product $A \star B$ of two $n \times n$ matrices A and B can be computed in $T(3n) \cdot \lceil \log_2(\max_{v,z \in G}\{\min_{u \in V}\{A_{vu} + B_{uv})\}\} \rceil$ rounds.

Proof idea:

- Use negative triangle detection to do a binary search for distances
- Observation: $A_{vu} + B_{uz} < d$ exactly whenever adding an edge of weight $-d$ from z to v gives a negative triangle → do a binary search like this!

Main References

- Elkin, M., Klauck, H., Nanongkai, D., & Pandurangan, G. (2014, July). Can quantum communication speed up distributed computation?. In *Proceedings of the 2014 ACM symposium on Principles of distributed computing* (pp. 166-175).
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